The influence of oscillations on natural convection in ship tanks

S. Doerffer and J. Mikielewicz*

The influence of low-frequency harmonic oscillations on the natural convection occurring at vertical walls of a ship tank has been investigated. The velocity field outside the boundary layer of the liquid in the tank has been determined by the use of potential flow theory. Two analytical models of heat transfer have been solved. The first of them is based on the method of small perturbations; while, in the second model, averaging of conservation equations is adopted. The models are valid for a laminar boundary layer. The results obtained agree well with experimental results obtained elsewhere[1]. Comparison has also been made, on the assumption that the laminar sublayer determines the heat transfer, with results of experimental investigations carried out by the authors for the turbulent flow range.

Keywords: *mixed convection, oscillations, ship tanks, heat transfer*

The design of heating systems mounted in a tank is based on knowledge of the principles of heat transfer between the liquid in the tank and the environment. For thermal calculations concerning a tank at rest, the commonly known principles of natural convection are adopted. No satisfactory methods have been found, however, for calculation of heat transfer in a tank subjected to oscillations. Holds of conventional tankers or oil bulk ore (obo) ships carrying crude oil or its products provide an example of such a type of tank.

In general, heat transfer in the holds of conventional tankers is laminar in nature. In such a case, calculations can be performed using the results of experimental investigations[1,2]. Until now, these have been the only investigations known to have dealt with the effect of ship rolling on the heat transfer between the walls of the tank and the fluid.

Unlike conventional tankers, new types of vessels, eg obo ships, have much larger holds with smooth walls or transverse framing. In such tanks, convection of decidedly turbulent nature occurs. Total lack of methods for calculating the heat transfer in the turbulent range during oscillations has been the reason for experimental investigations by one of the present authors[3]. Detailed knowledge of the problem is important since economic reasons necessitate precise design of heating systems that are to be installed on ships.

The general problem of heat transfer under conditions of stationary, turbulent natural convection superimposed by nonstationary inputs from the outside flow has yet to be solved analytically[4]. Also, no solutions have been available for a similar class of mixed convection problems where the outside flow is stationary[5]. On the other hand, there exist theoretical papers dealing with the influence of harmonic oscillations on the laminar boundary layer developing at a flat plate or cylinder[6-12]. Small oscillations are considered there, and solutions are most often limited to the first two approximations. The first one gives a solution for the stationary flow, ie for natural convection, while the second approximation consists of pulsatory components of hydromechanical and thermal quantities developed as a result of oscillations.

The lack of analytical methods for describing the heat transfer under conditions of the particular oscillatory flow which occurs at the walls of a ship tank, for the laminar as well as turbulent flow, has prompted the present authors to carry out their investigations.

Two theoretical models of heat transfer were developed, which allow determination of the pulsatory components of hydromechanical and thermal quantities as well as the influence of oscillations on the time-averaged heat transfer. The models were constructed for a laminar boundary layer; however, they can also be used for analysis of a turbulent boundary layer on the assumption that the laminar sublayer which is always present within a turbulent layer is mainly responsible for the thermal resistance. The analysis was carried out, without affecting the general character of the method, for a vertical wall of the tank, which is in the largest part responsible for the exchange of heat with the environment[1,2,13-16].

One type of forced motion of the tank was adopted: the harmonic rotation corresponding to the rolling of a ship. For fully loaded tankers, amplitudes of rolling are equal to several degrees, their period ranging from 5 s to 25 s. The rolling results in flow which is most intensive along the ship sides and the tank bottom[17,18]. To solve the models describing the heat transfer, knowledge of this flow is necessary.

Velocity field of liquid in a tank

The velocity field of liquid in an oscillating tank, and thus the velocity of washing its walls, were determined by the use of the theory of potential flow. The thickness of the boundary layer developing in the tank was estimated on the basis of the ship rolling parameters, tank size in the

* Institute of Fluid Flow Machinery of the Polish Academy of Sciences: Instytut Maszyn Przepływowych Polskiej Akademii Nauk, ul. Gen. J. Fiszera 14, 80-952 Gdańsk, Post Box 621, Poland
Received 9 August 1984 and accepted for publication in final form on 29 April 1985

range 10 m $\leqslant 2b \leqslant$ 40 m (Fig 1), and the viscosity of liquids carried and heated inside the tank, $\nu=(100–1500) \times 10^{-6}$ m^2/s. The results—the boundary layer thickness smaller by at least two orders of magnitude than the tank size—allowed us, following the Prandtl hypothesis[19], to apply the model of nonviscous fluid outside this layer and the potential flow theory to solution of the problem.

The analysis was carried out under the following assumptions:

a) The tank is a rigid rectangular structure, without inner structural members (Fig 1).
b) The liquid is nonviscous and incompressible, and fills the tank so that $l/2b>0.4$ (the condition of linear motion of the liquid surface during oscillations[20–23]).
c) An inertial reference system $OXYZ$, and a moving one $Oxyz$ coupled rigidly to the tank, as shown in Fig 1, are adopted.
d) The motion of the liquid is potential relative to $OXYZ$ and is two-dimensional, occurring in the plane OXZ.
e) Deformation of the free surface of the liquid from its mean position is small.
f) The surface tension is neglected.
g) The pressure at the free surface is atmospheric.
h) The forced motion of the tank is caused by harmonic input:

$$A=A_0 \sin \omega t \tag{1}$$

where the angular amplitudes A_0 are small and the frequencies ω are not close to the frequency ω_k of free oscillations of the liquid. The acceleration of the tank due to this motion is small relative to the acceleration due to gravity.

Free oscillations of the liquid

The liquid in the tank is a specific oscillating system. Therefore its free oscillations ought to be defined prior to the analysis of forced oscillations.

After a preliminary oscillation the tank remains immobile with respect to $OXYZ$. The assumptions as listed above allow solution of the problems on the basis of a linearized mathematical model[24]. A velocity potential function, $\varphi(x, z, t)$, is sought that would satisfy the Laplace equation in the entire volume occupied by the liquid:

$$\Delta\varphi(x, z, t)=0 \tag{2}$$

together with the boundary conditions:

i) at the tank walls

$$\left.\frac{\partial\varphi}{\partial x}\right|_{x=\pm b}=0$$
$$\left.\frac{\partial\varphi}{\partial z}\right|_{z=-l/2}=0 \tag{3}$$

ii) on the free surface of the liquid

$$\left.\frac{\partial^2\varphi}{\partial t^2}+g\frac{\partial\varphi}{\partial z}=0\right|_{z=l/2} \tag{4}$$

This leads to a formula for the frequency of free

Notation

a	Thermal diffusivity coefficient
A	Angle of inclination of the tank
b	Dimension of the tank, Fig 1; factor in Eq (41)
B	Function in Eq (37)
C	Factor in Eq (17)
e	Height of the tank, Fig 1
Fr	Froude number $=A_0\omega b/\sqrt{gl}$
Gr	Grashof number $=gl^3\beta\theta_w/\nu^2$
g	Gravitational acceleration
$I(\ )$	Imaginary part of complex number
i	Imaginary unit $=\sqrt{-1}$
l	Characteristic dimension; height of liquid in the tank
Nu	Nusselt number $=\alpha l/\lambda_f$
p	Pressure
Pr	Prandtl number $=\nu/a$
$R(\ )$	Real part of complex number
Ra	Reyleigh number $=PrGr$
Re	Reynolds number $=A_0\omega bl/\nu$
Sh	Strouhal number $=l/(A_0 b)$
t	Time
T	Temperature
T^*	Tank oscillation period
u, v, w	Components of velocity within the boundary layer
U	External flow velocity as defined by Eq (16)
x, y, z	Linear coordinates
Z^*	Position of the tank rotation axis, Fig 1
α	Heat transfer coefficient
β	Coefficient of volumetric expansion
Δ	Laplacian; boundary layer thickness ratio, δ_T/δ
δ	Boundary layer thickness
ε	Small parameter as defined by Eq (19)
η	Dimensionless coordinate as defined by Eq (28)
θ	Temperature difference
λ	Thermal conductivity
ν	Coefficient of kinematic viscosity
ρ	Density
φ	Velocity potential; phase shift
ω	Angular frequency of oscillations

Subscripts and superscripts

f	Liquid
k	Oscillation form
m	Mean value for the length, l
n	Term of a series
0	Stationary component; amplitude
r	Real part of a complex number
T	Thermal bounday layer
u	Imaginary part of a complex number
w	Wall
∞	Parameters outside the boundary layer
1	Pulsatory components
+	Dimensionless quantities
$\langle\ \rangle$	Quantities averaged over one oscillation period

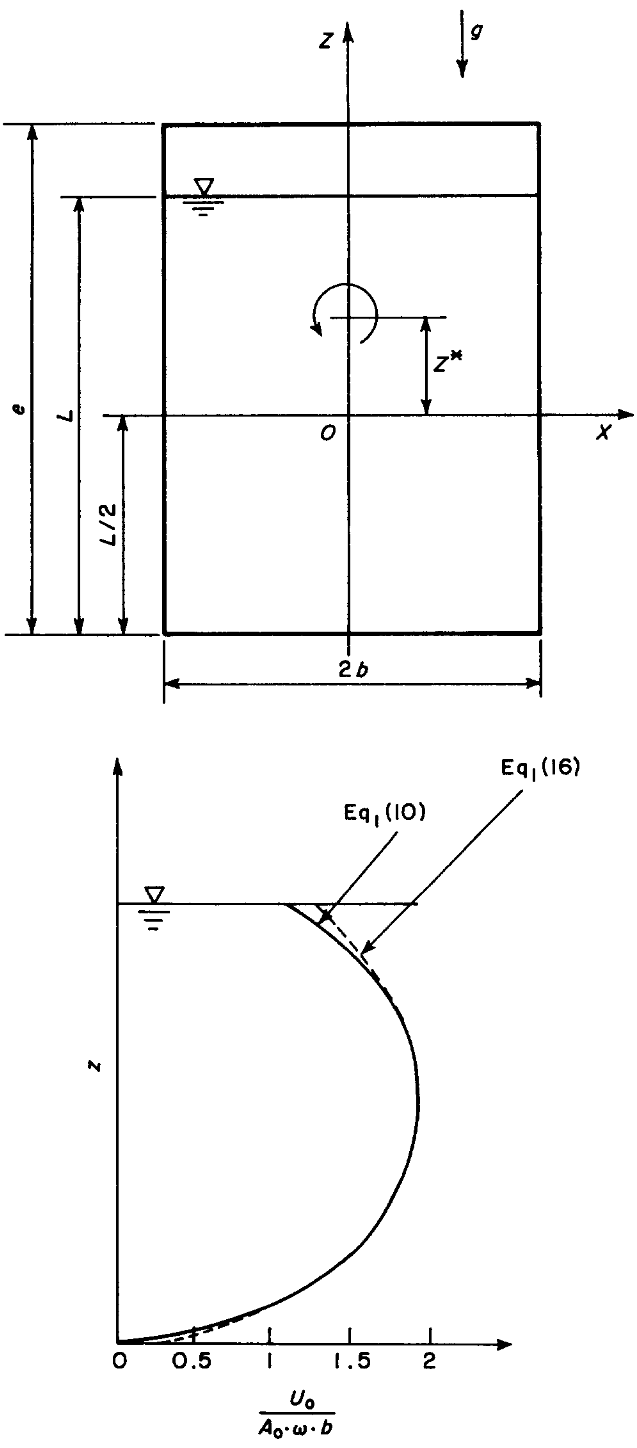

Fig 1 Model of tank, and the distribution of the relative velocity amplitudes at the vertical wall as related to the tangent velocity of the wall, $A_0\omega b$

oscillations of the liquid for all oscillation forms:

$$\omega_k = \sqrt{\frac{k\pi}{2b} g \, \mathrm{th}\left(\frac{k.\pi l}{2b}\right)}$$

$$(k = 1, 2, 3, \ldots) \tag{5}$$

The frequency is related to the oscillation period by the formula:

$$T_k^* = \frac{2\pi}{\omega_k} \tag{6}$$

$$(k = 1, 2, 3, \ldots)$$

The details of the solution can be found in Ref 25.

Forced oscillations

The tank oscillates about an axis of rotation passing through the point (O, Z^*). Its motion is described by Eq (1). The field of absolute velocities of the liquid defined with respect to $OXYZ$ is sought. If a linear model of the phenomenon is assumed and assumptions e) and h) above taken into consideration, then the velocity potential $\varphi(x, z, t)$ may be expressed directly in coordinates of the moving system $Oxyz$.

A velocity potential function $\varphi(x, z, t)$ is sought that satisfies the Laplace equation within the entire volume occupied by the liquid:

$$\Delta\varphi(x, z, t) = 0 \tag{7}$$

together with the boundary conditions

$$\left.\frac{\partial\varphi}{\partial x}\right|_{x=\pm b} = (z - Z^*) A_0 \omega \cos \omega t$$

$$\left.\frac{\partial\varphi}{\partial z}\right|_{z=-l/2} = x A_0 \omega \cos \omega t$$

$$\left.\frac{\partial^2\varphi}{\partial t^2} + g\frac{\partial\varphi}{\partial z} = 0\right|_{z=l/2} \tag{8}$$

Solution of Eq (7) together with conditions (8) by the method of Fourier series (for the details, see Ref 25) yields:

$$\varphi(x,z,t) = A_0\omega \cos \omega t \Bigg(-Z^* x + \sum_{k=1}^{\infty} \frac{(-1)^{k-1} 4l^2 \, \mathrm{sh}\left[(2k-1)\frac{\pi}{l} x\right]}{\pi^3 (2k-1)^3 \, \mathrm{ch}\left[(2k-1)\frac{\pi b}{l}\right]} \times$$

$$\times \sin\left[(2k-1)\frac{\pi z}{l}\right] + \sum_{k=1}^{\infty} \Bigg\{ \frac{(-1)^{k-1} 16 b^2}{\pi^3 (2k-1)^3} \times$$

$$\frac{\mathrm{ch}\left[(2k-1)\frac{\pi}{2b}\left(z - \frac{l}{2}\right)\right]}{\mathrm{sh}\left[(2k-1)\frac{\pi l}{2b}\right]} + \frac{(-1)^{k-1} 8b}{\pi^2 (2k-1)^2} \times$$

$$\frac{\omega^2}{(\omega_k^2 - \omega^2)} \frac{\mathrm{ch}\left[(2k-1)\frac{\pi}{2b}\left(z + \frac{l}{2}\right)\right]}{\mathrm{ch}\left[(2k-1)\frac{\pi l}{2b}\right]} \times \Bigg[-Z^* + \frac{l}{2} - \frac{4b \, \mathrm{th}\left[(2k-1)\frac{\pi l}{4b}\right]}{\pi (2k-1)} +$$

$$+ \frac{g}{\omega_k^2} \Bigg] \Bigg\} \sin\left[(2k-1)\frac{\pi}{2b} x\right] \Bigg) \tag{9}$$

The relations obtained above: (5), (6) and (9), lead to results analogous to those obtained from corresponding relations presented in Refs 22 and 26.

The relative velocity of the liquid with respect to the vertical wall of the tank is:

$$U = \left| -A_0 \omega b \cos \omega t - \left.\frac{\partial\varphi}{\partial z}\right|_{x=b} \right| \tag{10}$$

The amplitude of the velocity U divided by the tangent velocity of the vertical wall, $A_0\omega b$, versus the coordinate z, is shown in Fig 1.

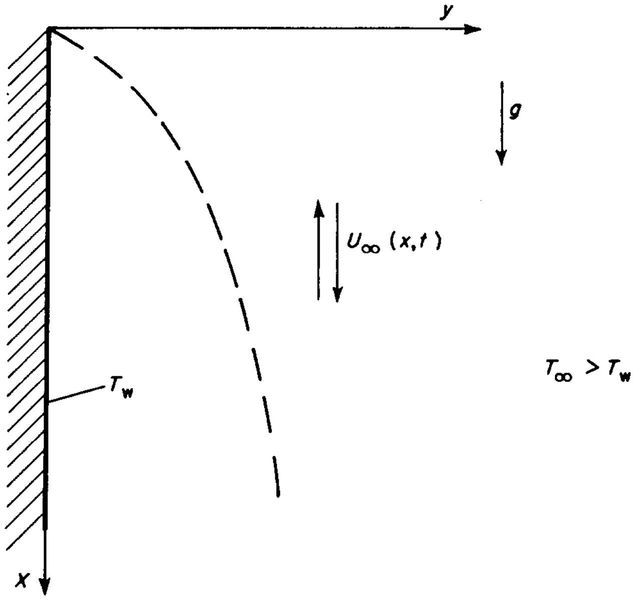

Fig 2 Sketch illustrating the mathematical model

The influence of oscillations on natural convection

The solution obtained within the framework of the theory of small perturbations

The motion due to natural convection perturbed by a forced flow in the form of small harmonic oscillations was analysed. The mathematical model of the phenomenon was formulated based on the following assumptions.

- A vertical flat plate situated as shown in Fig 2 is considered.
- The temperature of the plate, T_w, and that of the liquid outside the boundary layer, T_∞, is constant, with $T_\infty > T_w$.
- There is a laminar boundary layer, namely a hydraulic layer of thickness δ and a thermal layer of thickness δ_T.
- The liquid is incompressible and its properties do not vary; thermal changes of the density allowed for in the equation of motion are exceptions.
- The energy dissipation is neglected.
- The pressure gradient in the direction Oy equals zero:

$$\frac{\partial p'}{\partial y}=0.$$

- The velocity outside the boundary layer is $U_\infty(x,t)$ resulting from the potential motion of the liquid in the tank.
- The oscillatory motion of the tank is taken into account only through the velocity $U_\infty(x,t)$.

The assumptions adopted lead to the following equations of conservation:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g\beta\theta-\frac{1}{\rho}\frac{\partial p'}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \tag{11}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{12}$$

$$\frac{\partial \theta}{\partial t}+u\frac{\partial \theta}{\partial x}+v\frac{\partial \theta}{\partial y}=a\frac{\partial^2 \theta}{\partial y^2} \tag{13}$$

together with the boundary conditions:

$$y=0,\ u=v=0,\ \theta=\theta_w$$
$$y\to\infty,\ u=U_\infty(x,t),\ \theta=0 \tag{14}$$

where $\theta=T_\infty-T$.

In order to complete the set of Eqs (11) to (13) the pressure gradient $\partial p'/\partial x$ is determined from the condition satisfied at the layer boundary:

$$-\frac{1}{\rho}\frac{\partial p'}{\partial x}=\frac{\partial U_\infty}{\partial t}+U_\infty\frac{\partial U_\infty}{\partial x} \tag{15}$$

The velocity U as defined by Eqs (10) and (9) has a complex analytical form. A formula approximating this function has been adopted after a further analysis. In terms of Fig 1, and in the coordinate system as shown in Fig 2, it has the form:

$$U_\infty(x,t)=U_0(x)\cos\omega t \tag{16}$$

where the amplitude:

$$U_0(x)=A_0\omega b\left|2+C_1\,\mathrm{sh}\left[\frac{\pi}{2b}(l-x)\right]-C_2\,\mathrm{sh}\left(\frac{\pi x}{2b}\right)\right| \tag{17}$$

with

$$C_1=\frac{4\left[g-\omega^2\left(\frac{l}{2}-Z^*+\frac{2b}{\pi C_3}\right)\right]}{b\pi\left(\omega^2C_4-\frac{\pi g}{2b}C_3\right)}$$

$$C_2=\frac{16}{\pi^2C_3}$$

$$C_3=\mathrm{sh}\left(\frac{\pi l}{2b}\right)$$

$$C_4=\mathrm{ch}\left(\frac{\pi l}{2b}\right)$$

For a comparison of $U_\infty|_{max}$ according to Eqs (10) and (16), see Fig 1.

Eqs (11) to (13) together with conditions (14) form a complicated set of nonlinear partial differential equations. To make its solution possible, the set must be linearized. This may be done using the method of small parameters[19] under the assumption that the non-stationary forced flow, U_∞, is small compared with the stationary flow occurring due to natural convection.

The external flow may be written as:

$$U_\infty(x,t)=\varepsilon\tilde{U}_0(x)\cos\omega t \tag{18}$$

where $\varepsilon\in(0.1)$ is a small parameter defined as:

$$\varepsilon=\frac{U_0(x)}{u_0|_{max}} \tag{19}$$

while $u_0|_{max}$ is the maximum velocity of the convective motion at $x=l$.

It is useful to carry out the calculations for the flow $U_\infty(x,t)$ expressed in the complex form:

$$U_\infty(x,t)=\varepsilon\tilde{U}_0(x)\,e^{i\omega t} \tag{20}$$

It should be noted that a physical meaning may be assigned only to the real parts of Eq (20) and the solutions.

The solutions are sought in the form:

$$u(x,y,t)=u_0(x,y)+\varepsilon u_1(x,y)\,e^{i\omega t}+O(\varepsilon^2)$$
$$v(x,y,t)=v_0(x,y)+\varepsilon v_1(x,y)\,e^{i\omega t}+O(\varepsilon^2)$$
$$\theta(x,y,t)=\theta_0(x,y)+\varepsilon\theta_1(x,y)\,e^{i\omega t}+O(\varepsilon^2) \tag{21}$$

After substituting (21) into Eqs (11), (12), (13) and conditions (14), using Eqs (15) and (20), grouping together terms of the same order with respect to ε and neglecting those containing ε^2 and higher power of ε, one obtains two sets of differential equations and boundary conditions.

1) The first approximation—a set of equations describing the stationary heat transfer during natural convection—in a dimensionless form is:

$$u_0^+\frac{\partial u_0^+}{\partial x^+}+v_0^+\frac{\partial u_0^+}{\partial y^+}=Gr\,\theta_0^+ +\frac{\partial^2 u_0^+}{\partial y^{+2}}$$
$$\frac{\partial u_0^+}{\partial x^+}+\frac{\partial v_0^+}{\partial y^+}=0$$
$$u_0^+\frac{\partial \theta_0^+}{\partial x^+}+v_0^+\frac{\partial \theta_0^+}{\partial y^+}=\frac{1}{Pr}\frac{\partial^2 \theta_0^+}{\partial y^{+2}} \tag{22}$$

with the boundary conditions

$$y^+=0,\qquad u_0^+=v_0^+=0,\qquad \theta_0^+=1$$
$$y^+\to\infty,\qquad u_0^+=0,\qquad \theta_0^+=0 \tag{23}$$

2) The second approximation—a set of equations for the nonstationary heat transfer in an oscillatory flow—in a dimensionless form is:

$$Sh\,i(u_1^+-\tilde{U}_0^+)+u_0^+\frac{\partial u_1^+}{\partial x^+}+u_1^+\frac{\partial u_0^+}{\partial x^+}+v_0^+\frac{\partial u_1^+}{\partial y^+}+$$
$$+v_1^+\frac{\partial u_0^+}{\partial y^+}=\frac{Gr}{Re^2}\theta_1^+ -\frac{1}{Re}\frac{\partial^2 u_1^+}{\partial y^{+2}}$$
$$\frac{\partial u_1^+}{\partial x^+}+\frac{\partial v_1^+}{\partial y^+}=0 \tag{24}$$
$$Shi\theta_1^+ +u_0^+\frac{\partial \theta_1^+}{\partial x^+}+u_1^+\frac{\partial \theta_0^+}{\partial x^+}+v_0^+\frac{\partial \theta_1^+}{\partial y^+}+v_1^+\frac{\partial \theta_0^+}{\partial y^+}=$$
$$=\frac{1}{Re\,Pr}\frac{\partial^2\theta_1^+}{\partial y^{+2}}$$

together with the boundary conditions

$$y^+=0,\qquad u_1^+=v_1^+=0,\qquad \theta_1^+=0$$
$$y^+\to\infty,\qquad u_1^+=\tilde{U}_0^+,\qquad \theta_1^+=0 \tag{25}$$

where

$$x^+=\frac{x}{l},\qquad u_0^+=\frac{u_0}{(v/l)},\qquad u_1^+=\frac{u_1}{A_0\omega b}$$
$$\theta^+=\frac{\theta}{\theta_w}=\frac{T-T_w}{T_\infty-T_w},\qquad t^+=\frac{t}{(2\pi/\omega)}$$

For definitions of the remaining dimensionless numbers used in Eqs (22) and (24) see the Notation.

A solution of Eqs (22) with conditions (23) can be found in the literature. However, application of the method of balance equations for solution of (24) requires that solutions of (22) obtained by the same method be available.

Use of the continuity equation in the equations of motion and energy, and integration of Eqs (22) within the boundary layer thickness with conditions (23) leads to the balance equations:

$$\frac{\partial}{\partial x^+}\int_0^{\delta^+}u_0^{+2}\,dy^+=Gr\int_0^{\delta_T^+}\theta_0^+\,dy^+-\left.\frac{\partial u_0^+}{\partial y^+}\right|_{y^+=0}$$
$$\frac{\partial}{\partial x^+}\int_0^{\delta_T^+}u_0^+\theta_0^+\,dy^+=-\frac{1}{Pr}\left.\frac{\partial\theta_0^+}{\partial y^+}\right|_{y^+=0} \tag{26}$$

The following forms of the velocity and temperature profiles are assumed:

$$u_0^+=v^+\eta(1-\eta)^3$$
$$\theta_0^+=(1+\eta_T)(1-\eta_T)^3 \tag{27}$$

where

$$\eta=\frac{y^+}{\delta^+},\qquad \eta_T=\frac{y^+}{\delta_T^+} \tag{28}$$

Profiles (27) have been chosen so as to satisfy conditions (23) together with:

$$\left.\frac{\partial u_0^+}{\partial y^+}\right|_{y^+=\delta^+}=\left.\frac{\partial^2 u_0^+}{\partial y^{+2}}\right|_{y^+=\delta^+}=0$$
$$\left.\frac{\partial \theta_0}{\partial y^+}\right|_{y^+=\delta_T^+}=\left.\frac{\partial^2 \theta_0^+}{\partial y^{+2}}\right|_{y^+=\delta_T^+}=0$$
$$\left.\frac{\partial^2 u_0^+}{\partial y^{+2}}\right|_{y^+=0}+Gr=0 \tag{29}$$
$$\frac{1}{Pr}\left.\frac{\partial^2\theta_0^+}{\partial y^{+2}}\right|_{y^+=0}=0$$

For fluids characterized by $Pr\geqslant 1$, where the ratio $\Delta=\delta_T^+/\delta^+\leqslant 1$, the following solutions are obtained.

The local Nusselt number:

$$Nu_0=-\frac{1}{\delta_T^+}\left.\frac{\partial\theta_0^+}{\partial\eta_T}\right|_{\eta_T=0}=\frac{2}{\delta_0\Delta}x^{+(-1/4)} \tag{30}$$

The local thickness of the hydraulic boundary layer:

$$\delta^+=\delta_0x^{+(1/4)}=5.897\sqrt{\frac{(1.8\Delta-1)}{Gr}}\,x^{+(1/4)} \tag{31}$$

The distribution of the velocity component, u_0^+:

$$u_0^+=\frac{1}{6}Gr\delta_0^2x^{+(1/2)}\eta(1-\eta)^3 \tag{32}$$

The relation $\Delta=f(Pr)$ for $\Delta\leqslant 1$ assumes the form:

$$151.2\Delta(1.8\Delta-1)\left(\frac{1}{15}\Delta^2-\frac{1}{14}\Delta^3+\frac{9}{280}\Delta^4-\frac{1}{180}\Delta^5\right)=\frac{2}{Pr} \tag{33}$$

The solutions obtained correspond to those available in the literature for natural convection at a vertical wall.

The velocity given by Eq (32) should be expressed in a dimensionless form according to the principle adopted in Eqs (24); then it can be used for solving this set of equations. The dimensionless form of Eq (32) is:

$$u_0^+=\frac{1}{6}\frac{Gr}{Re}\delta_0^2x^{+(1/2)}\eta(1-\eta)^3 \tag{34}$$

On integration of the equations of motion and energy in set (24) in a manner analogous to that used for

(22) we obtain their balance form:

$$Sh\, i\int_0^1 u_1^+\,\mathrm{d}\eta-\tilde{U}_0^+\left[Sh\, i+\frac{1}{\delta^+}\frac{\partial}{\partial x}\left(\delta^+\int_0^1 u_0^+\,\mathrm{d}\eta\right)\right]+$$
$$+\frac{2}{\delta^+}\frac{\partial}{\partial x^+}\left[\delta^+\int_0^1 (u_0^+u_1^+)\,\mathrm{d}\eta\right]=\frac{Gr}{Re^2}\int_0^1\theta_1^+\,\mathrm{d}\eta-\frac{1}{\delta^{+2}Re}\frac{\partial u_1^+}{\partial\eta}\bigg|_{\eta=0} \tag{35}$$

$$Sh\, i\int_0^1 \theta_1^+\,\mathrm{d}\eta_T+\frac{1}{\delta_T^+}\frac{\partial}{\partial x^+}\left[\delta_T^+\int_0^1 (u_0^+\theta_1^++u_1^+\theta_0^+)\mathrm{d}\eta_T\right]=$$
$$=-\frac{1}{\delta_T^{+2}}\frac{1}{Re\,Pr}\frac{\partial\theta_1^+}{\partial\eta_T}\bigg|_{\eta_T=0} \tag{36}$$

The velocity and temperature profiles in the boundary layer have been assumed, as in Ref 7, to have the form:

$$u_1^+=\tilde{U}_0^+(5\eta^4-4\eta^5)+B(\eta-4\eta^4+3\eta^5)+$$
$$+\frac{1}{2}\delta^{+2}\,Re\,Sh\,\tilde{U}_0^+ i(-\eta^2+3\eta^4-2\eta^5)+$$
$$+\frac{1}{6}\delta^{+2}\,Sh\,Re\,iB(\eta^3-2\eta^4+\eta^5)+$$
$$+\frac{\delta^{+2}\,Gr}{6\Delta\,Re}B_1(-\eta^3+2\eta^4-\eta^5) \tag{37}$$

$$\theta_1^+=B_1[\eta_T-4\eta_T^4+3\eta_T^5+\frac{1}{6}\delta^{+2}\Delta^2\,Sh\,Re\,Pr\,i\times$$
$$\times(\eta_T^3-2\eta_T^4+\eta_T^5)] \tag{38}$$

chosen so as to ensure that conditions (25) be satisfied together with:

$$\frac{\partial u_1^+}{\partial\eta}\bigg|_{\eta=1}=0$$
$$\frac{\partial\theta_1^+}{\partial\eta_T}\bigg|_{\eta_T=1}=0$$
$$\frac{\partial^2 u_1^+}{\partial\eta^2}\bigg|_{\eta=0}=-\delta^{+2}\,Re\,Sh\,\tilde{U}_0^+ i \tag{39}$$
$$\frac{\partial^3 u_1^+}{\partial\eta^3}\bigg|_{\eta=0}=\delta^{+2}\,Re\,Sh\,i\frac{\partial u_1^+}{\partial\eta}\bigg|_{\eta=0}-\frac{\delta^{+2}Gr}{Re}\frac{\partial\theta_1^+}{\partial\eta}\bigg|_{\eta=0}$$
$$\frac{\partial^2\theta_1^+}{\partial\eta_T^2}\bigg|_{\eta_T=0}=0$$
$$\frac{\partial^3\theta_1^+}{\partial\eta_T^3}\bigg|_{\eta_T=0}=\delta_T^{+2}\,Sh\,Re\,Pr\,i\frac{\partial\theta_1^+}{\partial\eta_T}\bigg|_{\eta_T=0}$$

Substitution of Eqs (37), (38), (34) and (27) into Eqs (35) and (36) gives, on integration of these equations, a set of ordinary differential equations which are used, in turn, for determination of the complex functions $B(x^+)$ and $B_1(x^+)$ as introduced in Eqs (37) and (38). To this end the method of generalized series is adopted.

The local Nusselt number for this approximation has the following form:

$$Nu_{1x^+}=-\frac{1}{\delta_T^+}\frac{\partial\theta_1^+}{\partial\eta_T}\bigg|_{\eta_T=0}=-\frac{B_1(x^+)}{\delta_T^+(x^+)} \tag{40}$$

It is a complex number described by

$$R(Nu_{1x^+})=$$
$$=-\frac{1}{\delta_0\Delta}\sum_{n=0}^{\infty}b_{T(n)}[Sh\,Re\,\delta_0^2]^{(n-1)}x^{+[(2n-3)/4]}$$
$$I(Nu_{1x^+})=$$
$$=-\frac{1}{\delta_0\Delta}\sum_{n=0}^{\infty}b_{u(n)}[Sh\,Re\,\delta_0^2]^{(n-1)}x^{+[(2n-3)/4]} \tag{41}$$

The coefficients b_T and b_u are given in Ref 27.

Based on the temperature distribution in Eqs (21), the following relation for the local Nusselt number describing the heat transfer for the phenemonen under consideration is obtained:

$$Nu_{x^+,t^+}=Nu_{0x^+}+\varepsilon R(Nu_{1x^+}\exp i2\pi t^+)+0(\varepsilon^2) \tag{42}$$

where the second term can be written:

$$\varepsilon R(Nu_{1x^+}\exp i2\pi t^+)=\varepsilon|Nu_{1x^+}|\cos(2\pi t^++\varphi)$$

In Eq (42) the first term is the Nusselt number associated with the natural convection as described by Eq (30). The second term is a pulsatory component of the Nusselt number resulting from flow oscillations. It is given by the amplitude $\varepsilon|Nu_{1x^+}|$ and the phase shift φ relative to

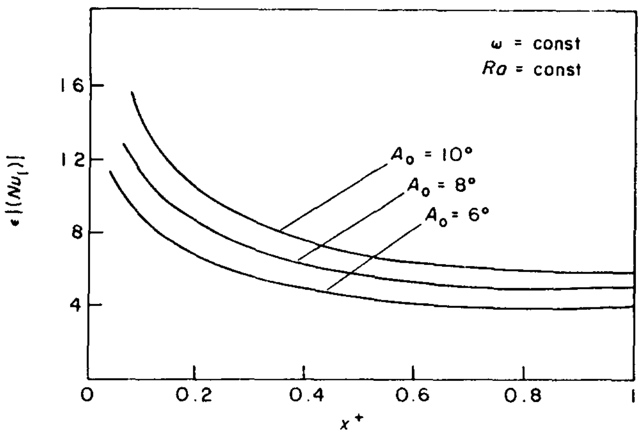

Fig 3 Distribution of the Nusselt number pulsatory component amplitudes for different tank oscillation amplitudes

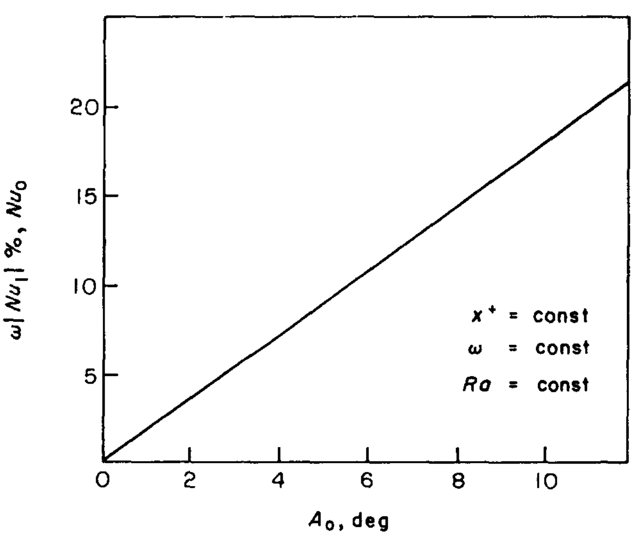

Fig 4 Ratio of amplitudes of the pulsatory and stationary components of the Nusselt number plotted against the tank oscillation amplitude

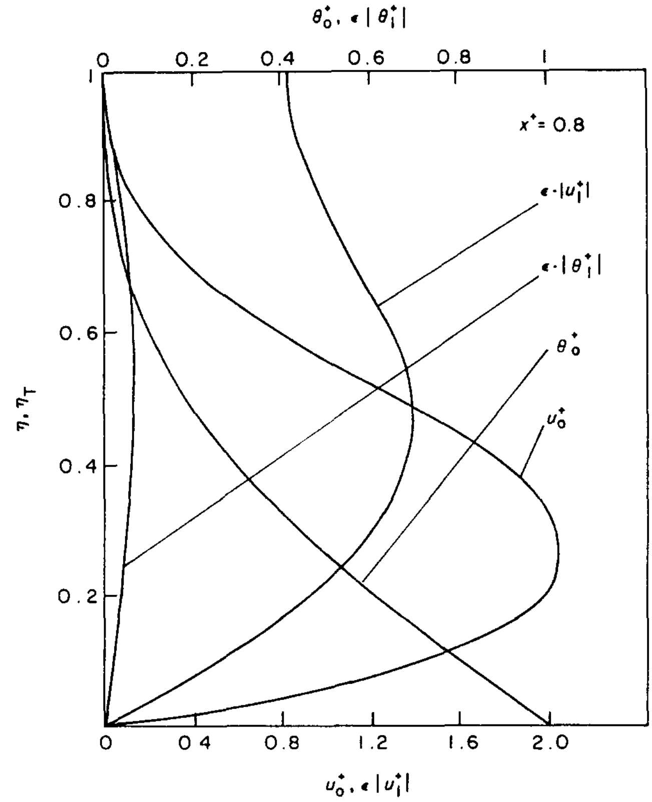

Fig 5 Velocity and temperature profiles in the boundary layer for $x^+=0.8$

the forced tank motion. This is illustrated in Fig 3, which shows the amplitudes $\varepsilon|Nu_{1x^+}|$ for different amplitudes of the forced motion, plotted against the coordinate x^+; and in Fig 4, showing the ratio of the pulsatory component amplitude to the stationary component Nu_{0x^+}. Fig 5 shows changes of the velocity and the temperature profiles in the boundary layer.

Solution by the averaging method

The model based on small perturbation theory yielded pulsatory components of the velocity and temperature developed as a result of oscillations. This section of the paper presents an attempt at estimating the effect of oscillations on the heat transfer averaged in time. To this end, averaged equations of conservation are used. The same assumptions as before are adopted with the exception that cases are taken into consideration for which the motion due to the forced flow of liquid in the tank prevails in the boundary layer over the natural convection flow.

The phenomenon is, therefore, described by the set of differential equations (11) to (13), and boundary conditions (14). In this case solutions are sought in the form:

$$\begin{aligned} u(x,y,t)&=u_0(x,y)+u_1(x,y,t)\\ v(x,y,t)&=v_0(x,y)+v_1(x,y,t)\\ \theta(x,y,t)&=\theta_0(x,y)+\theta_1(x,y,t) \end{aligned} \tag{43}$$

where the subscripts '0' denote the time-averaged values of the solutions, and '1' denote the pulsatory components. The mean values of the pulsatory components obtained for the oscillation period T^* are:

$$\langle u_1\rangle=\langle v_1\rangle=\langle\theta_1\rangle=0$$

Substitution of Eqs (43) into the set (11) to (13) and averaging over the period T^* gives a set of equations defining time-averaged components of the solutions. Equations for the pulsatory components are obtained by subtraction of the time-averaged equations from the complete ones[4,19].

The set of equations is very complex; therefore, the solutions are found in an approximative way. In the first approximation a time-averaged velocity component u_0 resulting from natural convection flow, is assumed to be independent from a pulsatory component u_1 due to the oscillations $U_\infty(x,t)$. Pulsatory component u_1 can be determined from the simplified momentum equation (the buoyancy and the convective terms are neglected[19]):

$$\frac{\partial u_1^+}{\partial t^+}=\frac{\partial U_\infty^+}{\partial t^+}+\frac{2\pi}{Sh\,Re}\frac{\partial^2 u_1^+}{\partial y^{+2}} \tag{44}$$

and the continuity equation:

$$\frac{\partial u_1^+}{\partial x^+}+\frac{\partial v_1^+}{\partial y^+}=0 \tag{45}$$

and boundary conditions:

$$\begin{aligned} &y^+=0, &&u_1^+=v_1^+=0\\ &y^+\to\infty, &&u_1^+=U_\infty^+ \end{aligned} \tag{46}$$

Solution of Eqs (44) and (45) together with conditions (46) gives:

$$u_1^+=U_0^+\left[\cos 2\pi t^+-(\exp-y^+/\delta^+)\cos\left(2\pi t^+-\frac{y^+}{\delta^+}\right)\right] \tag{47}$$

$$\begin{aligned} v_1^+=\frac{\mathrm{d}U_0^+}{\mathrm{d}x^+}\Bigg[&-y^+\cos 2\pi t^+ +\frac{1}{\sqrt{2}}\delta^+\cos\left(2\pi t^+-\frac{\pi}{4}\right)\\ &-\frac{\delta^+}{\sqrt{2}}\exp(-y^+/\delta^+)\cos\left(2\pi t^+-\frac{y^+}{\delta^+}-\frac{\pi}{4}\right)\Bigg] \end{aligned} \tag{48}$$

where

$$\delta^+=\frac{\delta}{l}=\frac{\sqrt{2}}{(Sh\,Re)^{1/2}}=\frac{1}{l}\sqrt{\frac{2\nu}{\omega}}$$

may serve as a measure of the thickness of the hydraulic boundary layer developing in the oscillatory flow along a flat plate (Fig 6).

For determination of the temperature field the following equations are used:

- for the pulsatory component, θ_1,

$$\begin{aligned} &\frac{\partial\theta_1}{\partial t}+u_1\frac{\partial\theta_0}{\partial x}+u_0\frac{\partial\theta_1}{\partial x}+u_1\frac{\partial\theta_1}{\partial x}+v_1\frac{\partial\theta_0}{\partial y}+\\ &\quad+v_0\frac{\partial\theta_1}{\partial y}+v_1\frac{\partial\theta_1}{\partial y}-\left\langle u_1\frac{\partial\theta_1}{\partial x}+v_1\frac{\partial\theta_1}{\partial y}\right\rangle=\\ &=a\frac{\partial^2\theta_1}{\partial y^2} \end{aligned} \tag{49}$$

- for the time-averaged component, θ_0,

$$u_0\frac{\partial\theta_0}{\partial x}+v_0\frac{\partial\theta_0}{\partial y}+\left\langle u_1\frac{\partial\theta_1}{\partial x}+v_1\frac{\partial\theta_1}{\partial y}\right\rangle=a\frac{\partial^2\theta_0}{\partial y^2} \tag{50}$$

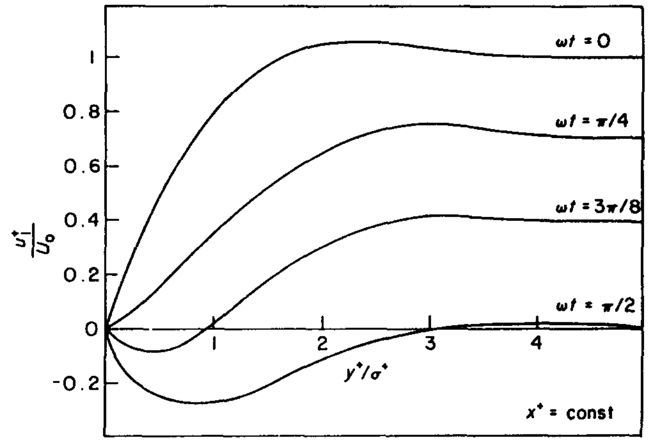

Fig 6 Distribution of the relative velocity u_1^+/U_0^+ *in the boundary layer*

The boundary conditions are:

$$\begin{aligned} &y=0, \quad \theta_0=\theta_w, \quad \theta_1=0 \\ &y=\delta_T, \quad \theta_0=0, \quad \theta_1=0 \end{aligned} \tag{51}$$

Approximate solution to this set of equations is sought. A time-averaged temperature profile is assumed:

$$\theta_0(x,y)=\theta_0'(x,y)+\theta_0''(x,y) \tag{52}$$

where θ_0' is the temperature responsible for natural convection and is independent from θ_0'' which describes the effect of oscillations.

Substituting Eq (52) into the energy equation (49) allows one to split Eq (49) into two independent equations. One of them, after linearization, can be written in the dimensionless form:

$$\frac{1}{2\pi} Sh \frac{\partial \theta_1^+}{\partial t^+} + u_1^+ \frac{\partial \theta_0'^+}{\partial x^+} + v_1^+ \frac{\partial \theta_0'^+}{\partial y^+} = \frac{1}{Re\,Pr} \frac{\partial^2 \theta_1^+}{\partial y^{+2}} \tag{53}$$

and can be solved for the conditions:

$$\begin{aligned} &y^+=0, \quad \theta_1^+=0 \\ &y^+=\delta_T^+, \quad \theta_1^+=0 \end{aligned} \tag{54}$$

This yields finally the pulsatory component of the Nusselt number:

$$Nu_1 = -\left.\frac{\partial \theta_1^+}{\partial y^+}\right|_{y^+=0} = f_1(Sh, Re, Pr, x^+, t^+) \tag{55}$$

The analytical function f_1 has a considerably complicated form[27].

When the pulsatory temperature component θ_1^+ is known then the time-averaged component θ_0^+ may be determined from Eq (50).

Application to Eq (50) of a procedure similar to that applied to Eq (49) gives the equation:

$$u_0 \frac{\partial \theta_0'}{\partial x} + v_0 \frac{\partial \theta_0'}{\partial y} = a \frac{\partial^2 \theta_0'}{\partial y^2} \tag{56}$$

while (51) gives the conditions:

$$\left.\begin{aligned} &y=0, \quad \theta_0'=\theta_w \\ &y=\delta_T, \quad \theta_0'=0 \end{aligned}\right\} \tag{57}$$

On the strength of assumptions adopted previously with regard to u_0 and v_0, the temperature distribution θ_0' developing in conditions of natural convection is just the solution of Eq (56) with conditions (57) as assumed in Eq (52).

The remaining terms of Eq (50), with less significant terms omitted, can be written in the dimensionless form:

$$\frac{1}{Re\,Pr} \frac{\partial^2 \theta_0''^+}{\partial y^{+2}} = \left\langle u_1^+ \frac{\partial \theta_1^+}{\partial x^+} + v_1^+ \frac{\partial \theta_1^+}{\partial y^+} \right\rangle \tag{58}$$

with boundary conditions:

$$\left.\begin{aligned} &y^+=0, \quad \theta_0''^+=0 \\ &y^+=\delta_T^+, \quad \theta_0''^+=0 \end{aligned}\right\} \tag{59}$$

The final solution to Eq (58) together with conditions (59) gives the stationary increase of the Nusselt number due to oscillations:

$$Nu_{0x^+}'' = -\left.\frac{\partial \theta_0''^+}{\partial y^+}\right|_{y^+=0} = f_2(Sh, Re, Pr, x^+) \tag{60}$$

The analytical function f_2 of complex form can be found in Ref 27.

For the approximations as specified above the Nusselt number characterizing the heat transfer under consideration is given by the expression:

$$Nu_{x^+,t^+} = Nu_{0x^+}' + Nu_{0x^+}'' + Nu_{1x^+,t^+} \tag{61}$$

The respective terms of formula (61) are given by Eqs (30), (60) and (55).

Fig 7 shows the stationary increase of the Nusselt number Nu_0'' and the amplitude of the pulsatory component $|Nu_1|$ plotted against the coordinate x^+.

Experimental verification of the model

The model, as presented in this paper, of mixed convection in a tank refers to laminar flow of liquid in the boundary layer. Results of experimental research[1] carried out in connection with laminar flow provide the best material for verification of the model. Our theoretical model has been formulated for small perturbations. It covers, therefore, only the lower region of the range investigated in Ref 1, which was specified to comply with shipbuilding needs. Table 1 shows a comparison of two results obtained for water and oil (runs No 1 and No 2 respectively).

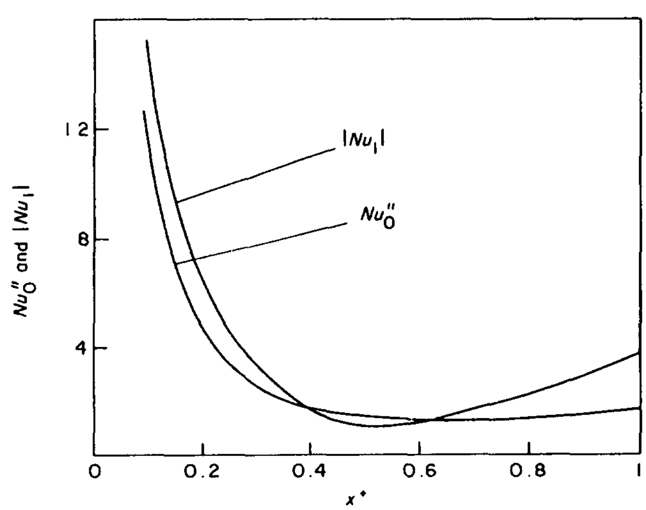

Fig 7 Stationary increase Nu_0'' *and the amplitude of the pulsatory component* $|Nu_1|$ *plotted against the coordinate* x^+

In Table 1, ΔNu_m is the mean increase of the Nusselt number Nu averaged in time and along the wall, over the number Nu_{0m} describing natural convection as obtained in investigations[1]. Nu''_{0m} is the same quantity as ΔNu_m, calculated by the method of averaging. The amplitudes of the pulsatory component Nu_{1m} calculated from the theory of small perturbations (model 1) and according to the method of averaging (model 2) are given in Table 2. It is evident that stationary increases of the Nusselt number obtained in the two ways are close to each other. This is not the case with values of amplitudes of the Nusselt number pulsatory components, as differences between results obtained based on the two theories can be as large as 25%.

To obtain data for turbulent heat transfer in the presence of oscillations, experiments were carried out by the authors[3]. To this end, a model tank of dimensions 0.5 m × 0.4 m × 0.3 m ($e \times 2b \times$ length as in Fig 1) made of organic glass plate 8 mm thick was used. The tank was in harmonic motion at amplitudes $A_0 = 0°$ to $10°$ and periods $T^* = 0.8$ s to 3.6 s about two axes of rotation located 0.19 m and 0.31 m above the bottom. A hydraulic driving installation was used. Distilled water and spindle oil ($\nu_{50°C} \cong 7.2 \times 10^{-6}$ m^2/s) were used for modelling the actual cargo. The measurements were carried out for two liquid heights: $L_1 = 0.32$ m and $L_2 = 0.2$ m. The heating system, comprising an electric heater rated 2000 W at 220 V, situated about 25 mm above the bottom, an autotransformer and a wattmeter, ensured the required thermal state of the liquid in the tank. The measurements were made for steady state thermal conditions and the turbulent character of heat transfer was maintained. The temperatures were measured with six sets of thermocouples situated along the vertical wall and the bottom of the tank. Each set consisted of three thermocouples, two of them mounted in the wall, on the inside and outside surfaces, and the third in the liquid relatively far from the wall.

Table 1 Comparison of Nusselt number steady increase between the experimental results of Ref 1 and the theory developed in this paper

Run No	Nu_{0m}	Experiment ΔNu_m	Theory Nu'_{0m}
1	14.54	0.672	0.677
2	18.92	0.258	0.255

The programme of investigations comprised:

- the examination of heat transfer in the absence of oscillations (ie free convection) to check the accuracy of the measuring technique and to obtain the reference data for further results (in the presence of oscillations);
- the examination of heat transfer in the presence of oscillations for various sets of the parameters.

About one hundred different cases were measured.

From an analysis of the experimental results (see Fig 8 for vertical wall) the natural and forced convection predominance regions were determined and the best correlation formulae were found. Figs 9, 10 and 11 show the experimental curves obtained for each region. The phenomenon was tested in the following ranges of dimensionless numbers:

$7 \times 10^7 \leqslant Ra \leqslant 3 \times 10^9$

$2.7 \leqslant Pr \leqslant 249$

$3.2 \times 10^2 \leqslant Re \leqslant 6 \times 10^4$

$5.7 \leqslant Sh \leqslant 36.7$

$1.1 \times 10^{-2} \leqslant Fr \leqslant 6.5 \times 10^{-2}$

The theoretical model described in this paper can not be used directly for analysis of turbulent heat transfer. It allows one, however, to carry out a simplified quantitative analysis of such cases. This is possible on the assumption that the total thermal resistance is concentrated in the laminar sublayer. This makes possible

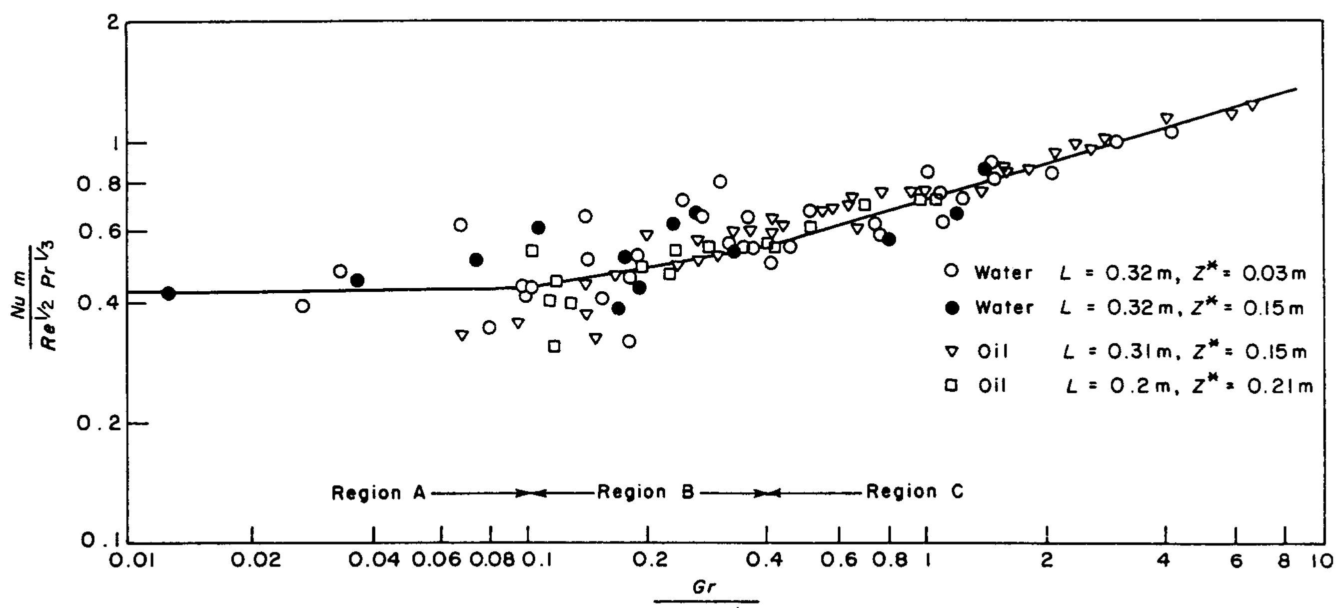

Fig 8 The generalized correlation for mixed convection at the vertical tank wall: A, the region of predominant forced convection for $Gr/(Re^2Pr^{1/3}) < 0.1$*; B, the region of mixed convection for* $0.1 \leqslant Gr/(Re^2Pr^{1/3}) \leqslant 0.4$*; C, the region of predominant natural convection for* $Gr/(Re^2 Pr^{1/3}) > 0.4$

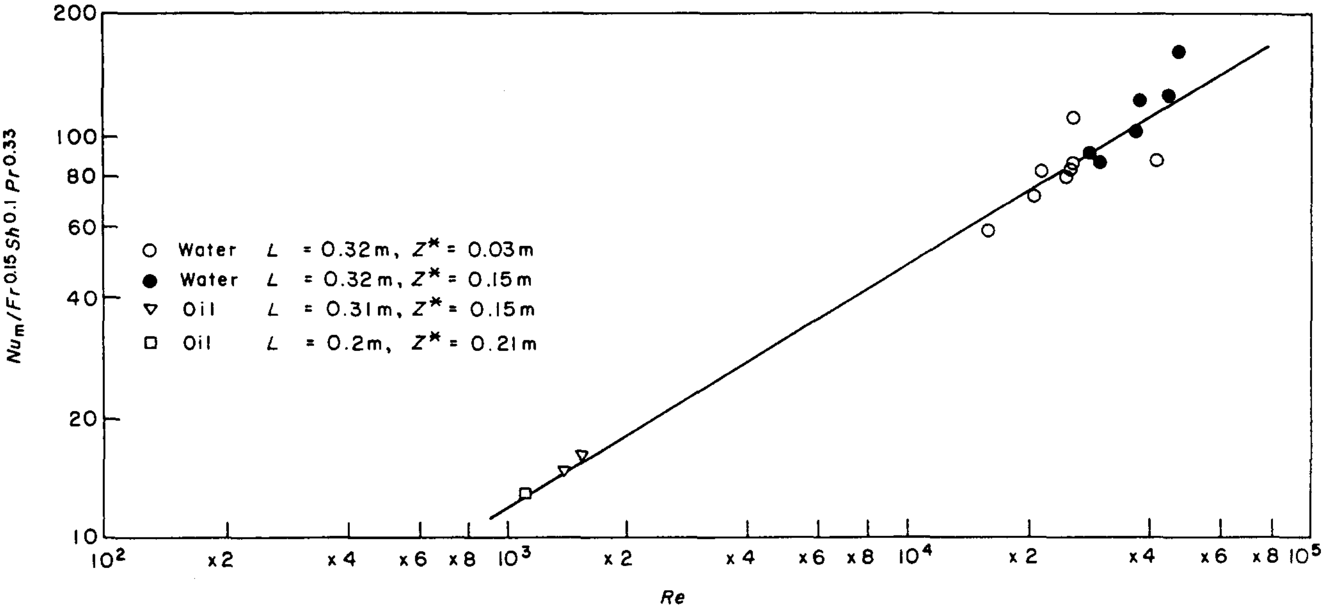

Fig 9 Heat transfer at the vertical wall in the region A described by: $\mathrm{Nu}_m = 0.185\,\mathrm{Re}^{0.605}\,\mathrm{Pr}^{1/3}\,\mathrm{Fr}^{0.15}\,\mathrm{Sh}^{0.1}$

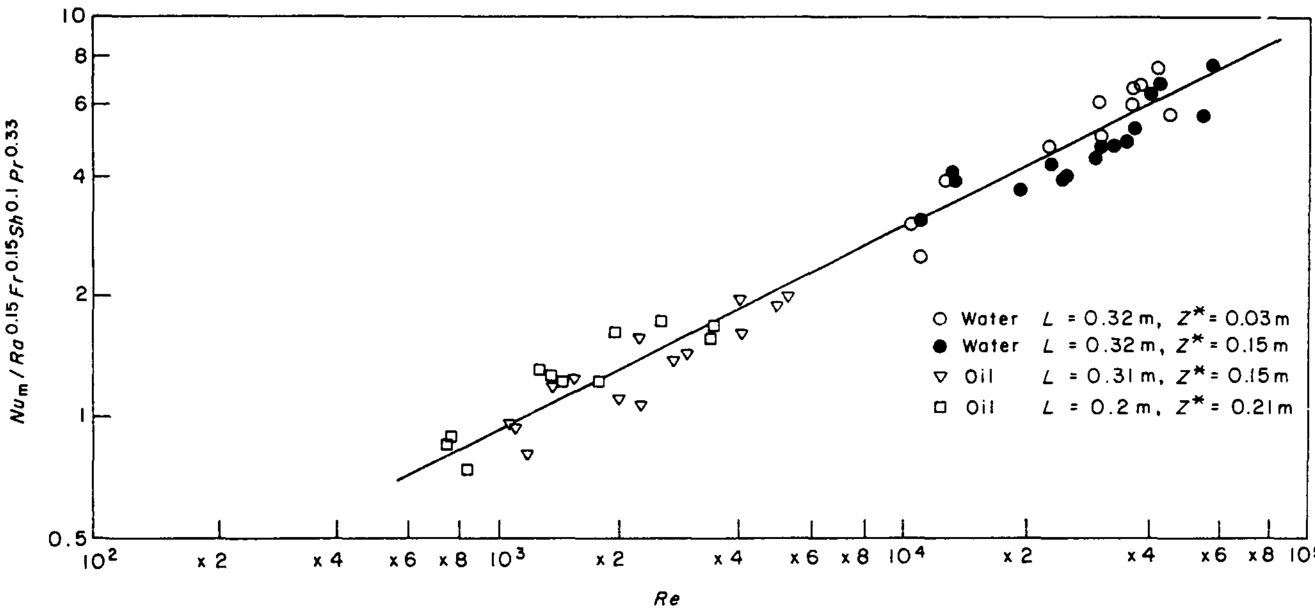

Fig 10 Heat transfer at the vertical wall in the region B described by: $\mathrm{Nu}_m = 2.88 \times 10^{-2}\,\mathrm{Re}^{0.5}\,\mathrm{Pr}^{1/3}\,\mathrm{Ra}^{0.15}\,\mathrm{Fr}^{0.15}\,\mathrm{Sh}^{0.1}$

Table 2 Comparison of values of the pulsatory component Nu_{1m} between theoretical models 1 and 2

Run No	Model 1	Model 2
1	1.481	1.334
2	0.528	0.659

estimation of its mean thickness:

$$\delta^+_{\mathrm{Tm}} = \frac{\delta_{\mathrm{Tm}}}{l} = \frac{\lambda_{\mathrm{f}}}{\alpha_{\mathrm{m}} l} = \frac{1}{Nu_{\mathrm{m}}}$$

where Nu_{m} is the mean value of the Nusselt number determined in the investigations above. Such an analysis was possible for only a few measuring points as parameters of the forced tank motion since the rest of them did not satisfy the assumptions of the theory. For the results see Table 3. A comparison of the Nusselt number Nu_{m} obtained by the averaging method and found experimentally gives fairly good agreement.

Discussion and conclusions

This paper concerns theoretical and experimental investigations of the complex phenomenon of heat transfer between a liquid and the walls of a tank subject to harmonic oscillations. The characteristic of the oscillatory flow of liquid in the tank has been determined on the basis of the theory of potential flow. Two theoretical models of the heat transfer under consideration were developed and solved approximately. Due to the complex nature of the mathematical models, and difficulties associated with their solution, only the cases of small perturbations and laminar flow in the boundary layer were taken into consideration. It would be very difficult to extend the

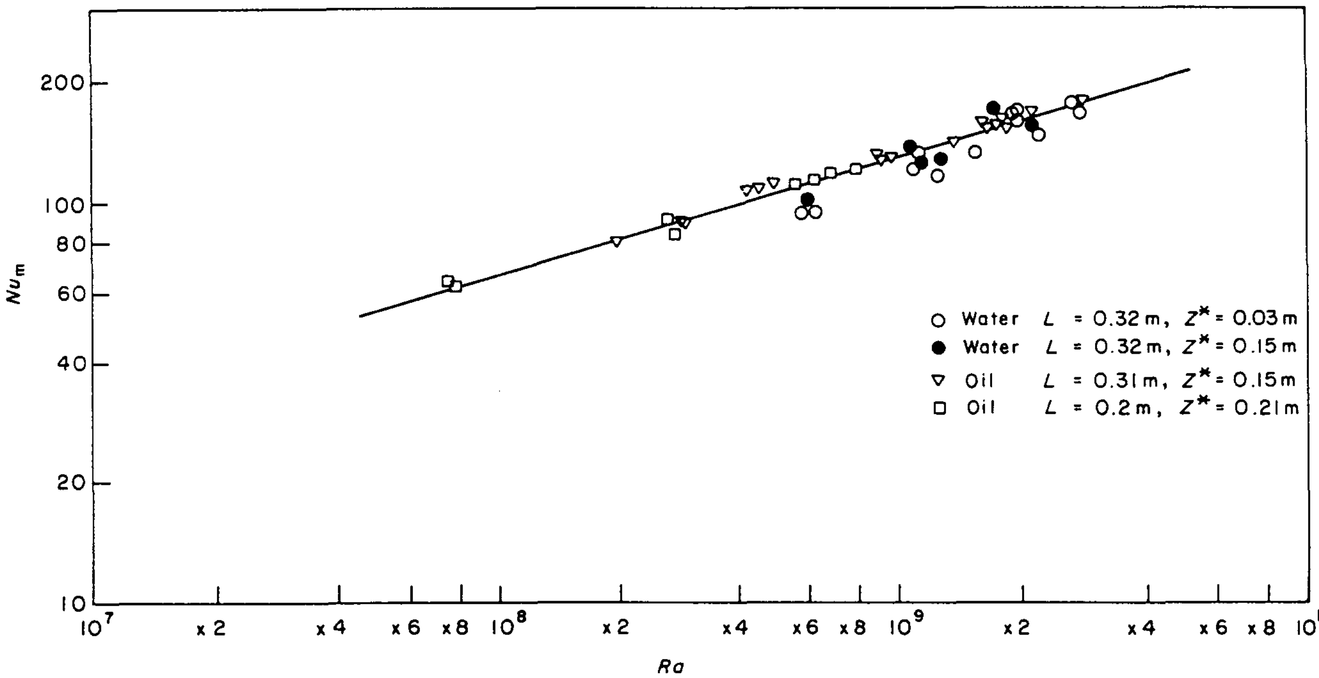

Fig 11 Heat transfer at the vertical wall in the region C described by: $Nu_m = 0.31\,Ra^{0.291}$

Table 3 Comparison of values of Nusselt number obtained from the authors' experiments and from the theory

Run no.	$\frac{Gr}{Re^2 Pr^{1/3}}$	Nu_{0m}	Experiment $Nu_m = Nu_{0m} + \Delta Nu_m$	Theory $Nu_m = Nu_{0m} + Nu''_{0m}$
1	3.081	161.7	157.8	162.1
2	0.398	117.9	119.6	120.6
3	0.099	104.9	115.2	108.7

theory to cover the case of large perturbations. So far no such theory is available.

Turbulent flow in the boundary layer can be analysed using the laminar models under the assumption that in turbulent flow only the laminar sublayer is responsible for the heat transfer mechanism.

The theories developed represent different approaches to the problem of oscillations in the boundary layer. In the first theory it is assumed that the flow occurring as a result of natural convection is the basic motion onto which oscillations are superimposed. In the second theory it is assumed that the predominant motion in the layer is that forced by the external flow. Besides, the temperature field corresponding to the case of natural convection is assumed for the initial iteration. Both theories provide good results in the case of small perturbations. However, each of them satisfies different needs of the user. From the first theory, amplitudes of the Nusselt number pulsatory component, velocities and temperatures as well as their phase shifts relative to the external inputs (useful in some applications) can be obtained. This allows one to complement, in a sense, experimental investigations by addition of velocity and temperature time courses, which are very difficult to measure and were not investigated in Refs 1 and 3. The other theoretical model is aimed mainly at determining time-averaged changes of the Nusselt number caused by oscillations. Amplitudes of the pulsations are less important. The discrepancies between amplitudes of pulsatory components, as shown in Table 2, result from different approximations characteristic to each theory. The qualitative character of changes along the wall is, however, the same; this is evident from Figs 3 and 7.

It was noticed that the dimensionless group $Gr/(Re^2\,Pr^{1/3})$ may be regarded as the principal criterion applicable for investigations of mixed convection. Its decrease corresponds to increasing participation of forced convection in this process (see Fig 8), which results in a rise of the stationary increase Nu''_0, thus demonstrating the validity of the theory (see Table 3).

The theories have been developed for one direction of the heat transfer, ie for flow of heat to the outside. A slight modification allows one to analyse also the case of heat input through the wall into the tank. The pattern of heat transfer is different in both cases, due to the nature of forced flow at the boundary layer border.

The theoretical models presented here, although they provide insight into the mechanism of mixed convection, are limited in their scope for practical applications to cases such as ship tanks. They cover only the narrow range of parameters met in these cases. Therefore, for design purposes, the experimental results of Ref 1 for laminar flow, and of the present paper for turbulent flow, can be applied.

References

1. **Kato H.** Effects of rolling on the heat transfer from cargo oil of tankers. *J. Soc. Naval Arch. Jap., 1969*, **126**, *421–430*
2. **Suhara J.** Studies of heat transfer on tank heating of tankers. *Jap. Shipp. a. Shipbuilding, 1970*, **5**(*1*), *5–16*
3. **Doerffer S.** *Results of Experimental Investigations of Heat Transfer in a Tank Subjected to Harmonic Oscillations. IF-FM Reports, 110/1007/81, 1981 (in Polish)*
4. **Galiciejski V. M.** *Tieplowye i Gidrodinamiceskie Procesy w Kolebajuscichsja Patokach. Masinostroenie, Moscow, 1977*
5. **Martynenko C. G.** *Tieploobmien Smiesannoj Konwekciej. Nauka i Technika, Minsk, 1975*
6. **Lighthill M. J.** The response of laminar skin friction and heat transfer to fluctuations in the stream velocity. *Proc. Roy. Soc., Series A, 1954*, **224**, *1–23*
7. **Eshghy S., Arpaci V. S. and Clark J. A.** The effect of longitudinal oscillations on free convection from vertical surfaces. *J. Appl. Mech., March 1965, 183–191*
8. **Schoenhals R. J. and Clark J. A.** Laminar free convection boundary layer perturbations due to transverse wall vibration. *J. Heat Transfer, August 1962, 225–234*
9. **Blankenship V. D. and Clark J. A.** Effects of oscillation on free convection from a vertical finite plate. *J. Heat Transfer, May 1964, 149–158*
10. **Nanda R. S. and Sharma V. P.** Free convection laminar boundary layers in oscillatory flow. *J. Fluid Mech., 1963*, **15**(*3*), *419–428*
11. **Stuart J. T.** A solution of the Navier–Stokes and energy equations illustrating the response of skin friction and temperature of an infinite plate thermometer to fluctuations in the stream. *Proc. Roy. Soc. Series A, 1955*, **231**, *116–130*
12. **Ishigaki H.** The effect of oscillation on flat plate heat transfer. *J. Fluid Mech., 1971*, **47**(*3*), *537–546*
13. **Akagi S.** Heat transfer in oil tanks of ship. *Jap. Shipbuild. a. Mar. Eng., 1969*, **4**(*2*), *26–35*
14. **Van der Heeden D. J.** Experimental evaluation of heat transfer in a dry-cargo ship's tank, using thermal oil as a heat transfer medium. *Int. Shipbuild. Prog., 1969*, **16**, *27–37*
15. **Suhara J.** Studies of heat transfer on tank heating of tankers. *Jap. Shipbuild. a. Mar. Eng., 1970*, **5**(*1*), *5–16*
16. **Saunders R. J.** Heat losses from oil-tanker cargoes. *Trans. Inst. Marine Engnrs, 1967*, **79**(*12*), *405–414*
17. **Abrahamsen E.** *Tank Size and Dynamic Loads on Bulk-Heads in Tankers. Det Norske Veritas, Publication No 28, 1962*
18. **Hunter, M., Dubois M. and Planeix J. M.** Model studies on the movement of liquid in tanks. *Marine Engnrs. Rev., January 1973, 25–28*
19. **Schlichting H.** *Boundary Layer Theory. McGraw-Hill Book Company, New York, 1968*
20. **Blixell A.** *Calculation of Wall Pressures in a Smooth Rectangular Tank due to Movement of Liquids. Lloyd's Register of Shipping, 1972, Report No 5108*
21. **Morris W. D. and Allen R. F.** Pressures in a tank due to sloshing. *Shipp. World a. Shipbuild., February 1974, 215–217*
22. **Hagiwara K. and Yamamoto Y.** A theory of sloshing in cargo oil tanks. *J. Soc. Naval Arch. Jap., 1962*, **112**, *143–152*
23. **Faltinsen O. M., Olsen H. A., Abramson H. N. and Bass R. L.** *Liquid Slosh in LNG Carriers. Det Norske Veritas, 1974, Publication No 85*
24. **Moisjejew N. N. and Pietrow A. A.** *Cislennye Metody Rasceta Sobstwiennych Castot Kolebanji Organicennogo Obema Zidkosti. Vicislitielnyj Centr. A.N. SSSR, Moscow, 1966*
25. **Doerffer S.** *Theoretical investigations of the behaviour of liquid in a ship's tank subjected to harmonic oscillations. IF-FM Report, 66/941/79, 1979 (in Polish)*
26. **Scarsi C. and Brizzolara E.** On the behaviour of liquid in a rectangular tank. *Int. Shipbuild. Prog., 1970*, **17**(*194*), *316–329*
27. **Doerffer S.** *The Problem of the Liquid Medium Oscillation Influence on Convective Heat Transfer as Applied to Ships' Tanks Carrying High Viscosity Liquids. Ph.D. Thesis, 1983, Thermodynamics and Heat Transfer Dept., Institute of Fluid Flow Machinery, P.A.Sci., Gdańsk (in Polish)*